STOCHASTIC INVISCID SHELL MODELS: WELL-POSEDNESS AND ANOMALOUS DISSIPATION.

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Abstract. In this paper we study a stochastic version of an inviscid shell model of turbulence with multiplicative noise. The deterministic counterpart of this model is quite general and includes inviscid GOY and Sabra shell models of turbulence. We prove global weak existence and uniqueness of solutions for any finite energy initial condition. Moreover energy dissipation of the system is proved in spite of its formal energy conservation.

Introduction

In recent years shell models of turbulence have attracted interest for their ability to capture some of the statistical properties and features of three-dimensional turbulence, while presenting a structure much simpler than Navier-Stokes and Euler equations.

The main idea behind shell models is to summarize in a unique variable u_n (usually complex-valued) all the modes with wave number k inside the shell $\lambda^n < |k| < \lambda^{n+1}$. Just like Navier-Stokes equations written in Fourier coordinates, the functions $\{u_n\}_{n\in\mathbb{N}}$ satisfy an infinite system of coupled ordinary equations, where the non-linear term is quadratic and formally preserves energy.

Shell models are however drastic modifications of Navier-Stokes equations. Firstly the variables $\{u_n\}_{n\in\mathbb{N}}$ representing three-dimensional shells (logarithmically equi-spaced) are one-dimensionally indexed by $\mathbb{N}$. Secondly the shells are allowed to interact only locally. The choice to allow only finite-range interactions is a crucial simplification both from analytical and numerical perspective but it is well justified inside the Kolmogorov theory of homogeneous turbulence, where one neglects energy exchanges between modes whose wave numbers differ for more than one order of magnitude.

These two characteristics of the shell models represent both their weakness and their strength. The main weaknesses are the loss of the geometry and the restriction to questions concerning the turbulence energy cascade only. The strengths are several: from a numerical perspective, the lower number of degrees of freedom allows for more accurate simulations at high Reynolds numbers (although the implementation of these simulations is not easy); from an analytical perspective, the simpler structure of the problem leads to sharper results both for the well-posedness and the understanding of the anomalous scaling exponents.

A review of the subject that focuses in particular on these aspects can be found in Biferale [12] which is devoted to the turbulence energy cascade and collects results concerning the structure function $S_p(k_n) = E[|u_n|^p]$ together with numerical evidence and analytical conjectures about anomalous exponents.

Main shell models. There are several different shell models in literature. The most studied are the GOY, introduced in Gledzer [22] and Ohkitani and Yamada [32] and the Sabra introduced in L'vov *et al.* [26]. Then there are two models with interactions that are somewhat simpler to study: one was introduced in Obukhov [28] and the other in Desnianskii and Novikov [20] and in Katz and

Pavlović [23]. While all of the previous have variables indexed by $\mathbb{N}$, there are also generalizations where the set of indexes is a regular tree (e.g. again in the first part of Katz and Pavlović [23] and in Barbato *et al.* [3]) which is closer to a true wavelet formulation of Navier-Stokes.

For the viscous versions of GOY and Sabra, well-posedness, global regularity of solutions and smooth dependence on the initial data are known (Constantin, Levant and Titi [17]). On the other hand, for the inviscid case less is known, the state of the art being Constantin, Levant and Titi [18] where the authors prove global existence of weak solutions and, for sufficiently smooth initial conditions, uniqueness and regularity for small times.

For the simpler shell models there are stronger results in the viscous case (see among the others Barbato, Morandin and Romito [9] and Cheskidov and Friedlander [13]), and even the inviscid case is understood quite well (the main results can be found in Katz and Pavlović [23], Kiselev and Zlatoš [24], Cheskidov, Friedlander and Pavlović [14] and [15], Barbato, Flandoli and Morandin [5] and [8] and Barbato and Morandin [4]).

Recently some stochastic shell models have been also proposed. An additive-noise version of the viscous GOY which is globally well-posed was introduced in Barbato *et al.* [2]. The existence of invariant measures was proved in Bessaih and Ferrario [10]. In Manna, Sritharan and Sundar [27] and Chueshov and Millet [16] a small multiplicative noise version of the GOY model is studied; well-posedness and a large deviation principle are established.

Finally, in Barbato, Flandoli and Morandin [6] and [7] a stochastic version of the inviscid Novikov model was proposed, which is then generalized to the tree-indexed Novikov model in Bianchi [11]. In these last models the noise term is multiplicative, and it is tailored to be formally energy-preserving. The cited papers prove global well-posedness of weak solutions for both models and anomalous dissipation for the former. (By anomalous dissipation we denote the property by which the total energy of the system decreases in spite of the formal conservativity of the dynamics.)

This type of noise is both elegant from an analytical point of view and physically meaningful in the sense that the interactions of Euler equations neglected in the shell models can be thought to be some sort of residual term which would behave (statistically) in a similar way.

Main results of the paper. In this paper we study a general stochastic inviscid shell model, with a multiplicative noise term similar to the one in Barbato, Flandoli and Morandin [6]. We restrict ourselves to indexes in $\mathbb{N}$, but we allow the variables X_n to be multidimensional.

This very general system of equations, given in (1), includes a stochastic version (51) of the inviscid GOY model (49) and a stochastic version (52) of the inviscid Sabra model (50).

The noise term is formally conservative, in the sense that it acts only on the transport of energy, without giving or taking any part of it. One general way to obtain such a noise is the following. Suppose that some shell model satisfies $\frac{d}{dt}u = B(u, u)$ with $Re\langle u, B(u, v)\rangle = 0$, then also the system $du = B(u, udt + \sigma \circ dW)$ will be formally conservative, the term $\sigma \circ dW$ being a perturbation acting only on the transport.

The first aim of this paper is to prove that for the general model (1) there are global weak existence and uniqueness in law of l^2 solutions. This result, to the knowledge of the authors, is the first result of global well-posedness for the inviscid GOY and Sabra models, both deterministic and stochastic.

The second important result concerns anomalous dissipation. Theorems 7 and 8 state that for both stochastic inviscid GOY and Sabra models energy decreases

with positive probability at all times, that it becomes arbitrarily small again with positive probability, and that if the initial energy is small enough, the solution converges to zero at least exponentially fast almost surely and in L^2.

Strategy and organization of the paper. Section 1 introduces and describes the general stochastic inviscid shell model (1). This is a subtle matter, since the requirement that the noise acts on the transport term in a conservative way leads to several algebraic conditions.

One of the key ideas of the paper is to use Girsanov theorem to study the original problem through an auxiliary linear system. Sections 2 and 3 are devoted to establish the relation between the linear and non-linear systems.

In Section 4 from the linear system we deduce the evolution equations for the second moment of components, which turn out to form a deterministic linear system and can be conveniently studied through the theory of q-matrices of continuous-time Markov chains. The first consequence is uniqueness of solutions: strong for the auxiliary linear system (Theorem 2), and in law for the main model (Theorem 3).

Existence of global solutions is classical and straightforward, and is detailed in Section 5. This concludes well-posedness for the main model.

Section 6 is devoted to anomalous dissipation, which is deduced from the behaviour of the continuous-time Markov chain associated to the equations for the second moment of components. In particular the chain turns out to be dishonest, in the sense that a.s. it reaches infinity in finite time. This "loss of mass" pulled back to the initial system becomes a loss of energy towards higher and higher components, which is what we call anomalous dissipation.

To formalize the link between the chain and the main model, one needs two steps: Borel-Cantelli lemma to get a.s. statements about the auxiliary linear system, and Novikov condition for Girsanov theorem to pass from the auxiliary system to the main model. Theorem 6 shows that if the initial energy is small with respect to the noise, Novikov condition holds also at $t = \infty$, and so even the exponential decay of energy can be deduced.

Finally we need to show that indeed GOY and Sabra models can be included in the general model (1) and summarize the results for these two models. This is done straightforwardly in Section 7.

1. Main model and formal requirements

The general model of this paper is equation (1) below. Since it is both complex and written in a synthetic but unfamiliar way, it will be helpful to start with a particular example to be kept in mind as a reference.

Consider on a complete filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$ the infinite system of stochastic differential equations in Stratonovich form below

$$dX_n = \lambda^{n-1} X_{n-1}^2 dt - \lambda^n X_n X_{n+1} dt \\ + \lambda^{n-1} X_{n-1} \circ dW_{n-1} - \lambda^n X_{n+1} \circ dW_n, \qquad n \geq 1,$$

where $(W_n)_{n\geq 0}$ is a sequence of independent Brownian motions and $X_0 \equiv 0$. This model is a stochastic version of the inviscid Novikov shell model, and was introduced in Barbato, Flandoli and Morandin [6] and [7]. The two deterministic terms are coupled in such a way to cancel when we sum $X_n dX_n$ over n. Apart form this, they are of the same form, representing an interaction between X_n and the product of two other components. The two stochastic terms are coupled among themselves in the same way, and moreover each of them is associated to one of the deterministic terms, so that the equation rewrites

$$dX_n = \lambda^{n-1} X_{n-1}(X_{n-1} dt + \circ dW_{n-1}) - \lambda^n X_{n+1}(X_n dt + \circ dW_n), \qquad n \geq 1.$$

We want to generalize this model to different types of interactions and multidimensional structure. We will also try to keep the notation as less cumbersome as possible, and in this view we will rewrite this as a sum over a set of interaction terms I, that will include pairs of cancelling interactions.

We are finally able to write the general model.

$$dX_n = \sum_{i\in I} k_{i,n} B_i(X_{n+r_i}, X_{n+h_i} dt + \sigma \circ dW_{i,n+h_i}), \qquad n \geq 1. \tag{1}$$

Here each X_n is a d-dimesional real-valued stochastic process, I is some finite set with an even number of elements and $(W_{i,n})_{i\in I, n\in\mathbb{Z}}$ is a family of d-dimensional brownian motions (independent apart from some deterministic relations explained below). For all $n \in \mathbb{Z}$ and $i \in I$, $k_{i,n}$ is a real constant, B_i is a bilinear operator on $\mathbb{R}^d$ while r_i and h_i are integer numbers.

In the example of Novikov model given above, I has two elements 1 and 2, $d = 1$, $B_i(a, b) = ab$ for $i = 1, 2$, the coefficients are given by

i	r_i	h_i	$k_{i,n}$
1	-1	-1	λ^{n-1}
2	1	0	$-\lambda^n$

and the Brownian motions are independent apart from $W_{1,n} = W_{2,n}$ a.s. for all n.

Going back to the general model, since r_i and h_i may be negative, we pose $X_n = 0$ for $n \leq 0$ and $k_{i,n} = 0$ for i, n such that $n + r_i \leq 0$ or $n + h_i \leq 0$. We will also require that $\overline{h} := \max_i h_i \geq 0$ otherwise X_n is constant for $n \leq -\overline{h}$.

We now list a first set of basic requirements on these models

i. *Finite range:* I is a finite set.
ii. *No self interactions:* $r_i \neq 0$ for all $i \in I$.
iii. *Exponential coefficients:* $k_{i,n} = \lambda^n k_i$ for all $i \in I_n$ and $n \geq 1$; here $\lambda > 1$ and k_i are real numbers and $I_n := \{i \in I : n + r_i \geq 1, n + h_i \geq 1\}$. If $i \notin I_n$ then $k_{i,n} = 0$.

The fourth but very important requirement is the *formal* (also called local) conservation of energy, which is assured by some cancellations, as described below. The intuitive meaning of the conditions detailed below is that I must be formed by pairs $\{i, \tilde{\imath}\}$ of cancelling interactions such that for all n there exists $\tilde{n} = n + r_i$ such that

$$k_{i,n} \langle X_n, B_i(X_{n+r_i}, X_{n+h_i} dt + \sigma \circ dW_{i,n+h_i})\rangle \\ + k_{\tilde{\imath},\tilde{n}} \langle X_{\tilde{n}}, B_{\tilde{\imath}}(X_{\tilde{n}+r_{\tilde{\imath}}}, X_{\tilde{n}+h_{\tilde{\imath}}} dt + \sigma \circ dW_{\tilde{\imath},\tilde{n}+h_{\tilde{\imath}}})\rangle = 0.$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$.

To make things formal and clean we need to introduce this definition.

Definition 1. Suppose τ is a permutation of I with no fixed point and such that $\tau = \tau^{-1}$. Let I^* be any subset of I such that I is the disjoint union of I^* and $\tau(I^*)$. We say that a family of d-dimensional Brownian motions $W = (W_{i,n})_{i\in I, n\in\mathbb{Z}}$ is symmetric with respect to τ if the restriction of W to $I^* \times \mathbb{Z}$ is a family of independent Brownian motions and $W_{\tau(i),n} = W_{i,n}$ a.s. for all $i \in I$ and $n \in \mathbb{Z}$.

Clearly this definition does not depend on the particular choice of I^*, but nevertheless by invoking this definition, we will implicitly suppose that we are fixing the set I^*.

We can now state the fourth requirement. We will use the notation $\tilde{\imath} = \tau(i)$.

iv. Local conservativity: there exists τ such that W is symmetric with respect to τ and the following relations hold for all $i \in I$

$$k_{\tilde{\imath}} = -k_i \lambda^{-r_i}, \tag{2}$$

$$\langle u, B_{\tilde{\imath}}(v,w)\rangle = \langle v, B_i(u,w)\rangle, \qquad \forall u,v,w \in \mathbb{R}^d, \tag{3}$$

$$r_{\tilde{\imath}} = -r_i, \tag{4}$$

$$h_{\tilde{\imath}} = h_i - r_i, \tag{5}$$

Remark 1. These algebraic requirements are meaningful, in the sense that given I and τ, there exist k, B, r and h satisfying them. Truly, it is easy to verify that however we define these objects on I^*, there is exactly one extension on all I satisfying the above conditions.

The following lemma summarizes some other trivial but useful consequences of the requirements above.

Lemma 2. *Let φ be the automorphism on $I \times \mathbb{Z}$ defined by $\varphi(i,n) := (\tilde{\imath}, \tilde{n}) := (\tau(i), n + r_i)$. Then there exists $\Delta \subset I \times \mathbb{Z}$ such that $\varphi(\Delta) = \Delta^c$. Moreover the following relations hold for all $i \in I$ and $n \in \mathbb{Z}$*

$$\begin{aligned} k_{\tilde{\imath},\tilde{n}} &= -k_{i,n} \\ \tilde{n} &= n + r_i \\ n &= \tilde{n} + r_{\tilde{\imath}} \\ W_{\tilde{\imath},\tilde{n}+h_{\tilde{\imath}}} &= W_{i,n+h_i} \quad a.s. \end{aligned}$$

In particular it is now straightforward that for all i and n,

$$\begin{aligned} k_{i,n} \langle X_n, B_i(X_{n+r_i}, X_{n+h_i} dt + \sigma \circ dW_{i,n+h_i})\rangle \\ + k_{\tilde{\imath},\tilde{n}} \langle X_{\tilde{n}}, B_{\tilde{\imath}}(X_{\tilde{n}+r_{\tilde{\imath}}}, X_{\tilde{n}+h_{\tilde{\imath}}} dt + \sigma \circ dW_{\tilde{\imath},\tilde{n}+h_{\tilde{\imath}}})\rangle = 0 \end{aligned} \tag{6}$$

Since by the Stratonovich form of Itô formula we have

$$d\langle X_n, X_n\rangle = 2\langle X_n, \circ dX_n\rangle,$$

if we sum formally these quantities over n substituting (1) and using (6), we have $\sum_{n\geq 1} d|X_n|^2 = 0$, so we may expect these models to be conservative.

Actually, this is in general not true. Rigorous arguments in the following sections will show that $d\sum_{n\geq 1} |X_n|^2 \neq \sum_{n\geq 1} d|X_n|^2$ and that $\sum_{n\geq 1} |X_n|^2$ decreases with positive probability.

2. Itō formulation and auxiliary equation

We prefer to reformulate equation (1) with Itō integration. Proposition 3 below states that the equivalent Itō differential equations are the following

$$dX_n = \sum_{i\in I} k_{i,n} B_i(X_{n+r_i}, X_{n+h_i} dt + \sigma dW_{i,n+h_i}) - \frac{\sigma^2}{2} \sum_{i \in I} k_{i,n}^2 L_i X_n dt, \qquad n \geq 1 \tag{7}$$

where L_i is the linear map on $\mathbb{R}^d$ given by $L_i := B_i B_i^T$. (Here B_i is interpreted as a linear map from $\mathbb{R}^{d^2}$ to $\mathbb{R}^d$.) In components, $L_i^{\alpha,\beta} = \sum_{\gamma,\delta} B_i^{\alpha,\gamma,\delta} B_i^{\beta,\gamma,\delta}$.

We will also introduce the auxiliary *linear* system of equations and their solutions. This will be needed afterwards, for Girsanov theorem.

$$dX_n = \sum_{i\in I} k_{i,n} B_i(X_{n+r_i}, \sigma dW_{i,n+h_i}) - \frac{\sigma^2}{2} \sum_{i\in I} k_{i,n}^2 L_i X_n dt, \qquad n \geq 1 \tag{8}$$

Let $H := l^2(\mathbb{R}^d)$ denote the state space and $\|\cdot\|$ its norm.

Definition 2. Given an initial condition $x \in H$, a weak solution of non-linear system (7) (respectively of linear system (8)) in H is a filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$, along with a family of Brownian motions W, and a stochastic process X such that

i. $W = (W_{i,n})_{i\in I, n\in\mathbb{Z}}$, is a family of d-dimensional Brownian motions on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$ adapted to the filtration and symmetric with respect to τ;

ii. $X = (X_n)_{n\geq 1}$ is an H-valued stochastic process on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$ with continuous adapted components;

iii. the following integral form of non-linear equation (7) holds for all $n \geq 1$ and all $t \geq 0$,

$$X_n(t) = x_n + \sum_{i\in I}\Big\{\int_0^t k_{i,n}B_i(X_{n+r_i}(s), X_{n+h_i}(s))ds + \int_0^t \sigma k_{i,n}B_i\big(X_{n+r_i}(s), dW_{i,n+h_i}(s)\big) - \int_0^t \frac{\sigma^2}{2}k_{i,n}^2 L_i X_n(s)ds\Big\} \tag{9}$$

(respectively, the following integral form of linear equation (8) holds for all $n \geq 1$ and all $t \geq 0$)

$$X_n(t) = x_n + \sum_{i\in I}\Big\{\int_0^t \sigma k_{i,n}B_i\big(X_{n+r_i}(s), dW_{i,n+h_i}(s)\big) - \int_0^t \frac{\sigma^2}{2}k_{i,n}^2 L_i X_n(s)ds\Big\} \tag{10}$$

The next proposition shows that what is defined above is actually a solution of the Stratonovich formulation of the non-linear system.

Proposition 3. *If X is a weak solution of the non-linear system (7), the Stratonovich integrals*

$$\int_0^t \sigma k_{i,n}B_i\big(X_{n+r_i}(s), \circ dW_{i,n+h_i}(s)\big) \tag{11}$$

are well defined and equal to

$$\int_0^t \sigma k_{i,n}B_i\big(X_{n+r_i}(s), dW_{i,n+h_i}(s)\big) - \int_0^t \frac{\sigma^2}{2}k_{i,n}^2 L_i X_n(s)ds \tag{12}$$

Hence X satisfies the Stratonovich equations (1).

Proof. We write components explicitly, in particular $B_i = \big(B_i^{\alpha,\beta,\gamma}\big)_{\alpha,\beta,\gamma}$ and $L_i^{\alpha,\beta} = \sum_{\gamma,\delta} B_i^{\alpha,\gamma,\delta}B_i^{\beta,\gamma,\delta}$. Component α of (11) rewrites

$$\sigma k_{i,n}\sum_{\beta,\gamma} B_i^{\alpha,\beta,\gamma}\int_0^t X_{n+r_i}^\beta(s)\circ dW_{i,n+h_i}^\gamma(s) \tag{13}$$

The stochastic integral can be rewritten in Itō form

$$\int_0^t X_{n+r_i}^\beta(s)\circ dW_{i,n+h_i}^\gamma(s) = \int_0^t X_{n+r_i}^\beta(s)dW_{i,n+h_i}^\gamma(s) + \frac{1}{2}\Big[X_{n+r_i}^\beta, W_{i,n+h_i}^\gamma\Big]_t \tag{14}$$

we only need to compute the quadratic covariation term. When $n+r_i\le 0$, this is zero, while if $n+r_i>0$, by writing (9) with $n+r_i$ in place of n, we find

$$
\begin{aligned}
(15)\quad &\Big[X^{\beta}_{n+r_i},W^{\gamma}_{i,n+h_i}\Big]_t\\
&=\Bigg[\sum_{j\in I}\sigma k_{j,n+r_i}\sum_{\delta,\eta}B_j^{\beta,\delta,\eta}\int_0^t X^{\delta}_{n+r_i+r_j}dW^{\eta}_{j,n+r_i+h_j},W^{\gamma}_{i,n+h_i}\Bigg]_t\\
&=\sigma\sum_{j\in I}k_{j,n+r_i}\sum_{\delta,\eta}B_j^{\beta,\delta,\eta}\int_0^t X^{\delta}_{n+r_i+r_j}(s)\;d\Big[W^{\eta}_{j,n+r_i+h_j},W^{\gamma}_{i,n+h_i}\Big]_s
\end{aligned}
$$

The very last quadratic covariation differential can be ds or 0, depending on whether the two particular BM involved are equal or independent. They are clearly independent when $j\notin\{i,\tilde{\imath}\}$. They are independent also when $j=i$, since $r_i\ne 0$. Finally, they are a.s. equal when $j=\tilde{\imath}$ and $\eta=\gamma$ by conditions (5) and (6). We get

$$
\begin{aligned}
(16)\quad \Big[X^{\beta}_{n+r_i},W^{\gamma}_{i,n+h_i}\Big]_t&=\sigma k_{\tilde{\imath},n+r_i}\sum_{\delta}B_{\tilde{\imath}}^{\beta,\delta,\gamma}\int_0^t X^{\delta}_{n+r_i+r_{\tilde{\imath}}}(s)ds\\
&=-\sigma k_{i,n}\sum_{\delta}B_i^{\delta,\beta,\gamma}\int_0^t X^{\delta}_n(s)ds
\end{aligned}
$$

where we used conditions (2), (3), and (4). Putting all together

(17)

$$
\begin{aligned}
&\Bigg[\int_0^t\sigma k_{i,n}B_i\big(X_{n+r_i}(s),\circ dW_{i,n+h_i}(s)\big)-\int_0^t\sigma k_{i,n}B_i\big(X_{n+r_i}(s),dW_{i,n+h_i}(s)\big)\Bigg]^{\alpha}\\
&=\frac{\sigma}{2}k_{i,n}\sum_{\beta,\gamma}B_i^{\alpha,\beta,\gamma}\Big[X^{\beta}_{n+r_i},W^{\gamma}_{i,n+h_i}\Big]_t=-\frac{\sigma^2}{2}k_{i,n}^2\sum_{\beta,\gamma}B_i^{\alpha,\beta,\gamma}\sum_{\delta}B_i^{\delta,\beta,\gamma}\int_0^t X^{\delta}_n(s)ds\\
&\quad=-\int_0^t\frac{\sigma^2}{2}k_{i,n}^2\sum_{\delta}X^{\delta}_n(s)ds\sum_{\beta,\gamma}B_i^{\alpha,\beta,\gamma}B_i^{\delta,\beta,\gamma}=-\Bigg[\int_0^t\frac{\sigma^2}{2}k_{i,n}^2L_iX_n(s)ds\Bigg]^{\alpha}
\end{aligned}
$$

The latter is correct also when $n+r_i\le 0$, since in that case $k_{i,n}=0$. □

Definition 3. Given an initial condition $x\in H$, an energy controlled solution of the non-linear system (7) or the linear system (8) is a weak solution of the same system of equations in the class $L^\infty(\Omega\times[0,\infty);H)$. In particular, if $\|X\|_{L^\infty}=\|x\|$, it is called a Leray solution.

3. Girsanov transformation

We turn our attention to the terms $X_{n+h_i}ds+\sigma dW_{i,n+h_i}$ in equation (7). We would like that, under a new probability measure Q, these were the differentials $\sigma dY_{i,n+h_i}$, where Y is again a family of d-dimensional Brownian motions symmetric with respect to τ. To do so, we state an infinite-dimensional version of Girsanov theorem whose proof can be found in Da Prato *et al.* [19].

Theorem 1. *On a filtered space $(\Omega,\mathcal{F}_\infty,(\mathcal{F}_t)_{t\ge0},P)$, let $(W_j)_{j\in\mathbb{N}}$ be a sequence of 1-dimensional adapted independent Brownian motions and let $X=(X_j)_{j\in\mathbb{N}}$ be a sequence of adapted semimartingales such that $\mathbb{E}\sum_j X_j^2(t)<\infty$ for all $t\ge0$.*

Let $Y_j(t):=\int_0^t X_j(s)ds+W_j(t)$ for $j\in\mathbb{N}$. Put, for $0\le t\le\infty$

$$M_t=\exp\Big\{-\int_0^t\sum_j X_j(s)dW_j(s)-\frac{1}{2}\int_0^t\sum_j X_j^2(s)ds\Big\}$$

Fix $0 < T \le \infty$. Suppose $\mathbb{E}[M_T] = 1$, then M is a closed martingale on $[0, T]$ and the density $\frac{dQ}{dP} = M_T$ defines a new probability measure Q on $\mathcal{F}_T$ under which $(Y_j)_{j\in\mathbb{N}}$ is a sequence of independent Brownian motions on $[0, T)$.

Moreover, to prove that $\mathbb{E}[M_T] = 1$, Novikov condition can be used, namely it is enough to prove that

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \sum_j X_j^2(s)ds\right\}\right] < \infty \tag{18}$$

By virtue of Theorem 1 it is quite easy to verify that, by changing the probability measure and the family of Brownian motions, any Leray solution of the non-linear system (7) can be transformed into a Leray solution of the auxiliary linear system (8). The reverse is also true.

In our notation, X will always denote the solution process; P and Q will denote the probability measures for the non-linear and linear systems respectively; P_T and Q_T their restrictions to $\mathcal{F}_T$; W and Y will denote the two associated families of Brownian motions.

The relation between W and Y is ensured by the following definition. For $i \in I$, $n \in \mathbb{N}$ and $t \ge 0$, let

$$Y_{i,n}(t) = \int_0^t \frac{1}{\sigma} X_n(s)ds + W_{i,n}(t) \tag{19}$$

Suppose we can define the two martingales

$$Z_t = -\int_0^t \sum_{\substack{i\in I^*\\ n\in\mathbb{N}}} \langle \sigma^{-1} X_{n+h_i}(s), dW_{i,n+h_i}(s)\rangle, \tag{20}$$

$$\tilde{Z}_t = \int_0^t \sum_{\substack{i\in I^*\\ n\in\mathbb{N}}} \langle \sigma^{-1} X_{n+h_i}(s), dY_{i,n+h_i}(s)\rangle \tag{21}$$

Then it easy to verify that

$$\tilde{Z}_t = -Z_t + \frac{1}{\sigma^2}\int_0^t \sum_{\substack{i\in I^*\\ n\in\mathbb{N}}} |X_{n+h_i}(s)|^2 ds = -Z_t + [Z, Z]_t = -Z_t + [\tilde{Z}, \tilde{Z}]_t$$

so that $Z_t - \frac{1}{2}[Z, Z]_t = -\tilde{Z}_t + \frac{1}{2}[\tilde{Z}, \tilde{Z}]_t$. We will then pose

$$\frac{dQ_t}{dP_t} = \exp\{Z_t - \frac{1}{2}[Z, Z]_t\}, \qquad \frac{dP_t}{dQ_t} = \exp\{\tilde{Z}_t - \frac{1}{2}[\tilde{Z}, \tilde{Z}]_t\} \tag{22}$$

We are now ready to make a precise statement

Proposition 4. *Suppose $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\ge0}, P, W, X)$ is a Leray solution of the non-linear system (7). Fix any $0 < T < \infty$. Let Q_T be the measure on $\mathcal{F}_T$ defined by (22) and let Y be the family of Brownian motions defined by (19).*

Then Q_T is a probability measure and $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\le t\le T}, Q_T, Y, X)$ is a Leray solution of the linear system (8) on $[0, T]$.

Proof. By (19), equation (9) is equivalent to equation (10) with W replaced by Y. We need to prove that Q_T is a probability measure and that Y is a family of Brownian motions symmetric w.r.t. τ, hence we apply Theorem 1.

The sequences we use are $\frac{1}{\sigma}X^j_{n+h_i}$ and $W^j_{i,n+h_i}$, both for $1 \le j \le d$, $n \in \mathbb{N}$ and $i \in I^*$. (Notice that we use I^* instead of I since we need the independence of the Brownian motions.)

By Leray property and the finiteness of I^*, we get a very strong bound

$$\sum_{i\in I^*}\sum_{n\geq 1}|X_{n+h_i}|^2(t)\leq |I^*|\|x\|^2 \qquad \text{a.s. for all } t>0 \tag{23}$$

by which we immediately deduce both that Z_t in (20) is well-defined and, by the finiteness of T, that

$$[Z,Z]_T=[\tilde{Z},\tilde{Z}]_T=\int_0^T\sum_{i\in I^*}\sum_{n\geq 1}\frac{1}{\sigma^2}|X_{n+h_i}|^2(s)ds\leq\frac{|I^*|\|x\|^2T}{\sigma^2}, \qquad a.s. \tag{24}$$

hence Novikov condition holds, namely

$$\mathbb{E}\big[e^{\frac{1}{2}[Z,Z]_T}\big]=\mathbb{E}\big[e^{\frac{1}{2}[\tilde{Z},\tilde{Z}]_T}\big]\leq\exp\frac{|I^*|\|x\|^2T}{2\sigma^2}<\infty \tag{25}$$

Finally, Y is symmetric w.r.t. τ since both P-a.s. and Q_T-a.s.

$$Y_{\tilde{\imath},n}(t)=\int_0^t\frac{1}{\sigma}X_n(s)ds+W_{\tilde{\imath},n}(t)=\int_0^t\frac{1}{\sigma}X_n(s)ds+W_{i,n}(t)=Y_{\tilde{\imath},n}(t) \tag{26}$$
□

The converse is also true. We give it without proof since it is almost identical to the one above.

Proposition 5. *Suppose $(\Omega,\mathcal{F}_\infty,(\mathcal{F}_t)_{t\geq 0},Q,Y,X)$ is a Leray solution of the linear system (8). Fix any $0<T<\infty$. Let P_T be the measure on $\mathcal{F}_T$ defined by the second one of (22) and let W be the family of Brownian motions defined by (19).*

Then P_T is a probability measure and $(\Omega,\mathcal{F}_T,(\mathcal{F}_t)_{0\leq t\leq T},P_T,W,X)$ is a Leray solution of the non-linear system (7) on $[0,T]$.

Remark 6. By Carathéodory theorem, the family of probability measures $(Q_T)_{T\geq 0}$ extends in a unique way to some probability measure Q on $\mathcal{F}_\infty$ and the same stands for $(P_T)_{T\geq 0}$. Hence solutions of non-linear and linear systems are associated also on infinite time span. From now on we will drop the $_T$ and use the symbols P and Q with this meaning. One should anyway keep in mind that while P_T and Q_T (and hence P and Q) are equivalent on $\mathcal{F}_T$ for any finite T, they are not in general equivalent on $\mathcal{F}_\infty$.

4. Closed equation for $\mathbb{E}^Q\left[|X_n(t)|^2\right]$ and uniqueness

Denote by $\mathbb{E}^Q$ the mathematical expectation on $(\Omega,\mathcal{F},Q)$. It turns out that if L_i is the identity for all $i\in I$, then $\mathbb{E}^Q\left[|X_n(t)|^2\right]$ satisfies a closed linear differential equation which will shed new light on the behaviour of solutions, in particular by giving an easy way to prove uniqueness.

Proposition 7. *Suppose $(\Omega,\mathcal{F}_\infty,(\mathcal{F}_t)_{t\geq 0},Q,Y,X)$ is an energy controlled solution of the linear system (8).*

Then for all $n\geq 1$ and $t\geq 0$

$$\frac{d}{dt}\mathbb{E}^Q\left[|X_n|^2\right]=-\sum_{i\in I}\sigma^2k_{i,n}^2\mathbb{E}^Q\langle X_n,L_iX_n\rangle+\sum_{i\in I}\sigma^2k_{i,n}^2\mathbb{E}^Q\langle X_{n+r_i},L_{\tilde{\imath}}X_{n+r_i}\rangle \tag{27}$$

Proof. We start by computing the quadratic variation of X_n. We use (10) and the independence of $Y_{i,n+h_i}$ and $Y_{j,n+h_j}$ when $j\neq i$. (If $j=\tilde{\imath}$, then $n+h_j=$

$n+h_i-r_i \neq n+h_i$.)

$$\begin{aligned}
\frac{d[X_n,X_n]_t}{\sigma^2} &= \sum_{i\in I}\sum_{j\in I} k_{i,n}k_{j,n}d\left[\int_0^t B_i(X_{n+r_i},dY_{i,n+h_i}),\int_0^t B_j(X_{n+r_j},dY_{j,n+h_j})\right]_t \\
&= \sum_{i\in I} k_{i,n}^2 d\left[\int_0^t B_i(X_{n+r_i},dY_{i,n+h_i}),\int_0^t B_i(X_{n+r_i},dY_{i,n+h_i})\right]_t \\
&= \sum_{i\in I} k_{i,n}^2 \sum_{\alpha,\beta,\gamma,\delta,\epsilon} B_i^{\alpha,\beta,\gamma}B_i^{\alpha,\delta,\epsilon}X_{n+r_i}^\beta X_{n+r_i}^\delta d[Y_{i,n+h_i}^\gamma,Y_{i,n+h_i}^\epsilon]_t \\
&= \sum_{i\in I} k_{i,n}^2 \sum_{\beta,\delta}\sum_{\alpha,\gamma} B_{\tilde{i}}^{\beta,\alpha,\gamma}B_{\tilde{i}}^{\delta,\alpha,\gamma}X_{n+r_i}^\beta X_{n+r_i}^\delta dt \\
&= \sum_{i\in I} k_{i,n}^2 \langle X_{n+r_i},L_{\tilde{i}}X_{n+r_i}\rangle dt
\end{aligned}$$

We also used (3) and the definition of $L_i^{\alpha,\beta}=\sum_{\gamma,\delta}B_i^{\alpha,\gamma,\delta}B_i^{\beta,\gamma,\delta}$.

Now we are able to compute the differential of $|X_n(t)|^2$

$$\begin{aligned}
d(|X_n(t)|^2) &= 2\langle X_n(t),dX_n(t)\rangle + d[X_n,X_n]_t \\
&= 2\sum_{i\in I}\sigma k_{i,n}\langle X_n,B_i(X_{n+r_i},dY_{i,n+h_i})\rangle - \sum_{i\in I}\sigma^2k_{i,n}^2\langle X_n,L_iX_n\rangle dt+ \\
&\qquad + \sum_{i\in I}\sigma^2k_{i,n}^2\langle X_{n+r_i},L_{\tilde{i}}X_{n+r_i}\rangle dt
\end{aligned} \tag{28}$$

By the definition of energy controlled solution, $|X_n(t)|\le\|X(t)\|\le C$ almost surely, so in particular $\mathbb{E}^Q\int_0^t|X_n|^2(s)|X_{n+r_i}|^2(s)ds<\infty$ and hence the first term above is a martingale, with mathematical expectation zero. If we now take the mathematical expectation of the integral form of (28) we get the integral form of (27) and we are finished. □

When all the L_i are the identity, equation (27) becomes a linear closed differential equation

$$\frac{d}{dt}\mathbb{E}^Q\left[|X_n|^2\right] = -\sum_{i\in I}\sigma^2k_{i,n}^2\mathbb{E}^Q\left[|X_n|^2\right]+\sum_{i\in I}\sigma^2k_{i,n}^2\mathbb{E}^Q\left[|X_{n+r_i}|^2\right] \tag{29}$$

We notice that this system of equations is of a peculiar kind, with negative diagonal and non-negative off-diagonal entries, thus suggesting a connection to the Kolmogorov equations for continuous-time Markov chains on the positive integers. We investigate this relation presently.

Denote by $\Pi=(\pi_{m,n})_{m,n\ge1}$ the infinite matrix associated to this system: for $n,m\ge1$ and $m\neq n$ let

$$\begin{cases}
\pi_{n,m} := \displaystyle\sum_{\substack{i\in I\\ r_i=m-n}}\sigma^2k_{i,n}^2 = \sigma^2\lambda^{2n}\sum_{\substack{i\in I_n\\ r_i=m-n}}k_i^2 \\
\pi_{n,n} := -\displaystyle\sum_{i\in I}\sigma^2k_{i,n}^2 = -\sigma^2\lambda^{2n}\sum_{i\in I_n}k_i^2 =: -\pi_n
\end{cases} \tag{30}$$

Remark 8. Recall that for $n\ge1$, $I_n:=\{i\in I: n+r_i\ge1, n+h_i\ge1\}$, so that

$$I_1\subset I_2\subset\cdots\subset I_{n_0}=I_{n_0+1}=\cdots=I.$$

where $n_0\ge 1-\min\{r_i,h_i:i\in I\}$. Hence for example $\pi_n=O(\lambda^{2n})$ as $n\to\infty$.

Corollary 9. *Suppose that* (Q, X) *is an energy controlled solution of equation* (8) *and that all the* L_i *are the identity. Then* $u = (u_n)_{n\geq 1}$ *defined by* $u_n(t) = \mathbb{E}^Q\left[|X_n(t)|^2\right]$ *is a non-negative solution of the Cauchy problem*

$$\begin{cases} u' = u\Pi \\ u_n(0) = |x_n|^2 \quad n \geq 1 \end{cases} \tag{31}$$

in the class $L^\infty\big([0,\infty); l^1\big)$.

Proposition 10. *The infinite matrix* Π *defined above is the stable, conservative q-matrix of a continuous-time Markov chain on the positive integers. Moreover* Π *is symmetric.*

Proof. Π is a stable q-matrix if $\pi_{n,n} < 0$ for all n, $\pi_{n,m} \geq 0$ whenever $n \neq m$ and for all n

$$\sum_{m: m\neq n} \pi_{n,m} \leq \pi_n, \tag{32}$$

moreover it is conservative if the latter holds with equality.

The first two conditions are obvious, and the third one follows from the fact that the sets $\{i \in I : n + r_i = m\}$ with $m \geq 1$, $m \neq n$ form a partition of $\{i \in I : n + r_i \geq 1, k_{i,n} \neq 0\}$.

Finally, for $m \neq n$,

$$\pi_{n,m} = \sum_{\substack{\tilde{\imath}\in I \\ r_i = m-n}} \sigma^2 k_{\tilde{\imath},n}^2 = \sum_{\substack{i\in I \\ r_i = n-m}} \sigma^2 k_{i,n-r_i}^2 = \sum_{\substack{i\in I \\ r_i = n-m}} \sigma^2 k_{i,m}^2 = \pi_{m,n} \tag{33}$$ □

The q-matrix of a continuous-time Markov chain is associated to the forward and backward Kolmogorov equations, namely

$$u' = u\Pi \tag{34}$$

$$u' = \Pi u \tag{35}$$

The transition probabilities of the Markov chain $p_{n,m}(t)$ solve both equations, in the classes l^1 and l^∞ respectively, with fixed n and m respectively and with initial condition $u_{n,m}(0) = \delta_{n,m}$.

These equations always have at least one shared "special" solution $f_{i,j}(t)$, which is a transition function, and is called the minimal solution. They do not always have uniqueness of solutions. Here it will be important that there is uniqueness for the forward equation and not for the backward. The key information is that the q-matrix is symmetric.

Lemma 11. *Suppose* Π *is a stable and* symmetric *q-matrix. Consider the forward equations with zero initial condition. Then the only non-negative solution in* $L^\infty([0,\infty); l^1)$ *is zero.*

More in general, given any non-negative l^1 *initial condition, there a unique solution in the same class.*

Proof. For the first part, we follow the classical approach by Laplace transform, introduced by Feller [21]. Let ρ be such a solution. For all $n \geq 1$ and $t \geq 0$ we have

$$\begin{cases} \rho_n'(t) = \sum_k \rho_k(t)\pi_{k,n} \\ \rho_n(t) \geq 0 \\ \rho_n(0) = 0 \\ \sum_k \rho_k(t) \leq C \end{cases}$$

For all $n \geq 1$, let $z_n = \int_0^\infty e^{-t}\rho_n(t)dt$. Clearly $\sum_n z_n \leq C$, so we can choose m such that $z_m \geq z_k$ for all k.

Notice that, since Π is stable and symmetric

$$|\rho'_m(t)| = |-\pi_m\rho_m(t) + \sum_{k\neq m}\pi_{m,k}\rho_k(t)| \leq \pi_m\rho_m(t) + \pi_m C \leq 2C\pi_m < \infty$$

hence we can integrate by parts and use symmetry and stability again to get

$$\begin{aligned} z_m = \int_0^\infty e^{-t}\rho'_m(t)dt &= \int_0^\infty e^{-t}\sum_k \rho_k(t)\pi_{k,m}dt = \sum_k z_k\pi_{k,m} \\ &= -z_m\pi_m + \sum_{k\neq m}\pi_{m,k}z_k \leq -z_m\pi_m + z_m\sum_{k\neq m}\pi_{m,k} \leq 0 \end{aligned}$$

We conclude that $z_n = 0$ and $\rho_n \equiv 0$ for all n.

For the general case, let $f_{i,j}(t)$ be the minimal solution of Π and let u^0 be a non-negative, l^1 initial condition. Then $u_n(t) = \sum_{i\geq 1} u_i^0 f_{i,n}(t)$ is a solution in the required class. Let v be another such solution and let $\rho = v - u$. By a forward integral recursion (FIR) approach it is easy to show that the minimality of f passes to u, in that $v_n(t) \geq u_n(t)$. (See for example Anderson [1], Theorem 2.2.2.) So ρ is a solution of the same problem, but with null initial condition and the first part of the lemma applies. ☐

We have finally collected all elements to prove uniqueness of solutions.

Theorem 2. *Suppose L_i is the identity matrix for all $i \in I$. Then there is strong uniqueness for the linear system (8) in the class of $L^\infty(\Omega \times [0,\infty); H)$ solutions.*

Proof. By linearity of (8) it is enough to prove that when the initial condition is $x = 0$ there is no non-trivial solution. Suppose (Q, X) is any energy controlled solution with zero initial condition, then by Corollary 9, Proposition 10 and the first part of Lemma 11, $\mathbb{E}^Q\left[|X_n(t)|^2\right] = 0$ for all n and t, hence $X = 0$ a.s. ☐

Remark 12. This result applies seamlessly also to the case of L^∞, non-anticipative, random initial conditions.

Uniqueness of solutions for the auxiliary linear system is then inherited by the original non-linear system, but in a weakened form.

Theorem 3. *Suppose L_i is the identity matrix for all $i \in I$ and let $T > 0$. Then there is uniqueness in law for the non-linear system (7) in the class of Leray $L^\infty(\Omega\times [0,T]; H)$ solutions.*

Proof. Suppose we are given two solutions $(P^{(1)}, W^{(1)}, X^{(1)})$ and $(P^{(2)}, W^{(2)}, X^{(2)})$. We want to prove that

$$\begin{aligned} &\mathbb{E}^{P^{(1)}}[f(X^{(1)}(t_1), X^{(1)}(t_2), \dots, X^{(1)}(t_n))] \\ &\qquad = \mathbb{E}^{P^{(2)}}[f(X^{(2)}(t_1), X^{(2)}(t_2), \dots, X^{(2)}(t_n))] \end{aligned} \tag{36}$$

where f is any bounded measurable real function on H^n and $t_1, t_2, \dots, t_n \in [0,T]$. By Proposition 4 and the first one of (22) we have that, for $j = 1, 2$

$$\begin{aligned} &\mathbb{E}^{P^{(j)}}[f(X^{(j)}(t_1), X^{(j)}(t_2), \dots, X^{(j)}(t_n))] \\ &\qquad = \mathbb{E}^{Q^{(j)}}[\exp\{-Z_T^{(j)} + \frac{1}{2}[Z^{(j)}, Z^{(j)}]_T\} f(X^{(j)}(t_1), X^{(j)}(t_2), \dots, X^{(j)}(t_n))] \end{aligned} \tag{37}$$

Where $Z^{(j)}$ is defined by (20). By Theorem 2 equation (8) has strong uniqueness, hence we can apply an infinite-dimensional version of Yamada-Watanabe theorem (see Revuz and Yor [31] or Prévôt and Röckner [30]) to deduce that the laws of

$(X^{(1)}, W^{(1)})$ and $(X^{(2)}, W^{(2)})$ on $C([0,T];\mathbb{R}^{2d})^{\mathbb{N}}$ are equal, under $Q^{(1)}$ and $Q^{(2)}$ respectively.

Then of course we can include also $Z^{(i)}$ by their definition and conclude that $(X^{(1)}, W^{(1)}, Z^{(1)})$ under $Q^{(1)}$ and $(X^{(2)}, W^{(2)}, Z^{(2)})$ under $Q^{(2)}$ have the same law, yielding in particular that (37) does not depend on j. $\square$

5. Existence of solutions

In this section we prove strong existence of solutions for the linear auxiliary model and deduce weak existence for the non-linear model. The approach is by finite-dimensional approximation and follows Pardoux [29] and Krylov and Rozovskiĭ [25]. The border term of the finite-dimensional systems is chosen so that energy conservation holds, giving a strong tool to prove convergence.

Theorem 4. *Let $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, Q)$ be a filtered probability space. Let Y be a family of adapted d-dimensional Brownian motions symmetric with respect to τ. Given an initial condition $\chi \in L^\infty((\Omega, \mathcal{F}_0); H)$ and $T > 0$, there exists at least an H-valued stochastic process X with continuous adapted components, such that Q-almost surely $\|X(t)\| \leq \|\chi\|$ for all $t \geq 0$ and for all $n \geq 1$ and all $t \geq 0$,*

(38)
$$X_n(t) = \chi_n + \sum_{i\in I}\Big\{\int_0^t \sigma k_{i,n} B_i\big(X_{n+r_i}(s), dW_{i,n+h_i}(s)\big) - \int_0^t \frac{\sigma^2}{2} k_{i,n}^2 L_i X_n(s) ds\Big\}$$

Such a process is called strong Leray solution.

Proof. For every positive N, let $A_N = \{1, \dots, N\}$ and consider the finite dimensional stochastic linear system

(39)
$$\begin{cases} dX_n^{(N)} = \sum_{i\in I} k_{i,n} B_i(X_{n+r_i}^{(N)}, \sigma dY_{i,n+h_i}) - \frac{\sigma^2}{2}\sum_{i\in I} k_{i,n}^2 L_i X_n^{(N)} dt, & n \in A_N \\ X_n^{(N)}(0) = \chi_n, & n \in A_N \\ X_n^{(N)} \equiv 0, & n \in \mathbb{Z}\setminus A_N \end{cases}$$

This system has a unique global strong solution $X^{(N)}$. By the local conservativity, we can prove that the l^2 norm is Q-a.s. constant. In particular we notice that equation (28) applies to $X_n^{(N)}$, for $n \in A_N$ without modifications. Then we sum on n and apply Lemma 2 to get

$$\begin{aligned}\sum_{n\in A_N} d(|X_n^{(N)}(t)|^2) &= 2\sum_{I\times\mathbb{Z}} \sigma k_{i,n}\big\langle X_n^{(N)}, B_i(X_{n+r_i}^{(N)}, dY_{i,n+h_i})\big\rangle \\ &\quad - \sum_{I\times\mathbb{Z}} \sigma^2 k_{i,n}^2 \big\langle X_n^{(N)}, L_i X_n^{(N)}\big\rangle dt + \sum_{I\times\mathbb{Z}} \sigma^2 k_{i,n}^2 \langle X_{n+r_i}^{(N)}, L_{\tilde{\imath}} X_{n+r_i}^{(N)}\rangle dt \\ &= 2\sigma\sum_{\Delta}\Big\{k_{i,n}\big\langle X_n^{(N)}, B_i(X_{n+r_i}^{(N)}, dY_{i,n+h_i})\big\rangle + k_{\tilde{\imath},\tilde{n}}\big\langle X_{\tilde{n}}^{(N)}, B_{\tilde{\imath}}(X_{\tilde{n}+r_{\tilde{\imath}}}^{(N)}, dY_{\tilde{\imath},\tilde{n}+h_{\tilde{\imath}}})\big\rangle\Big\} \\ &\quad - \sum_{I\times\mathbb{Z}} \sigma^2 k_{i,n}^2 \big\langle X_n^{(N)}, L_i X_n^{(N)}\big\rangle dt + \sum_{I\times\mathbb{Z}} \sigma^2 k_{\tilde{\imath},\tilde{n}}^2 \langle X_{\tilde{n}}^{(N)}, L_{\tilde{\imath}} X_{\tilde{n}}^{(N)}\rangle dt = 0\end{aligned}$$

Thus

$$\sum_{n\in A_N} |X_n^{(N)}(t)|^2 = \sum_{n\in A_N} |\chi_n|^2 \leq \|\chi\|^2_{L^\infty(\Omega;H)}, \qquad \forall t \geq 0, \quad Q\text{-a.s.} \tag{40}$$

meaning in particular that the sequence $X^{(N)}$ is bounded in $L^\infty(\Omega \times [0,T]; H)$. Hence there exists X in the same space and a sequence $N_k \uparrow \infty$ such that $X^{(N_k)} \xrightarrow{w*} X$ as $k \to \infty$. A fortiori there is also weak convergence in $L^2(\Omega \times [0,T]; H)$.

Let $\mathbb{X}$ denote the subspace of $L^2(\Omega \times [0,T]; H)$ of the progressively measurable processes, then $X^{(N)} \in \mathbb{X}$ for all N. The space $\mathbb{X}$ is complete, hence it is a closed subspace of L^2 in the strong topology, hence it is also closed in the weak topology, so X must be progressively measurable.

Now we want to show that X indeed satisfies equation (10) with Y in place of W, as each of the $X^{(N)}$'s does. Fix $t \in [0,T]$, $i \in I$, $n \geq 1$ and $l, m \in \{1, 2, \dots, d\}$. The map

$$V \to \int_0^t V^l_{n+r_i}(s) dY^m_{i,n+h_i}(s)$$

is a linear strongly continuous map from $\mathbb{X}$ to $L^2(\Omega; \mathbb{R})$, so it is weakly continuous. Equation (10), written component-wise, reduces to a finite sum of one-dimensional stochastic integrals like the one above, hence we can pass to the limit and so X solves the same equations. A posteriori, from these integral equations, it follows that there is a modification such that all components are continuous.

Finally we prove the Leray property. Consider the product measure $\mu = Q \times \mathcal{L}$ on $\Omega \times [0,T]$. Let $\epsilon > 0$, let $A = \{(\omega,t) : \|X(\omega,t)\| \geq \|\chi(\omega)\| + \epsilon\}$ and let $U = \frac{X}{\|X\|}\mathbb{1}_A \in L^2(\Omega \times [0,T]; H)$. Then

$$\langle X, U \rangle_{L^2} = \int \|X\| \mathbb{1}_A d\mu \geq \int_A \|\chi\| d\mu + \epsilon \mu(A)$$

On the other hand by Cauchy-Schwartz inequality on H and by (40).

$$\langle X^{(N_k)}, U \rangle_{L^2} = \int \left\langle X^{(N_k)}, \frac{X}{\|X\|} \right\rangle_H \mathbb{1}_A d\mu \leq \int \|X^{(N_k)}\| \mathbb{1}_A d\mu \leq \int_A \|\chi\| d\mu$$

Taking again the weak limit, we have $\mu(A) = 0$ and by the arbitrarity of ϵ, we get that μ-a.e. $\|X\| \leq \|\chi\|$. This can be improved by the continuity of components, which implies that the maps $t \mapsto \sum_{k \leq n} |X_k(t)|^2$ are continuous and hence Q-a.s. bounded for all t and all n by $\|\chi\|^2$. Letting $n \to \infty$ we conclude. □

The strong existence statement for the linear model becomes a weak existence statement for the non-linear model, due to Proposition 5.

Corollary 13. *Given an initial condition $x \in H$ and $T > 0$, there exists at least one Leray solution of the non-linear system (7) in the class $L^\infty(\Omega \times [0,T]; H)$.*

6. Anomalous dissipation

In this section we want to prove that $\|X(t)\|$ goes to zero in some sense. We consider the differential equation for the second moments (29) and study the continuous-time Markov chain that has it as its forward Kolmogorov equation. The following proposition gives an explicit connection between the two.

Proposition 14. *Suppose L_i is the identity matrix for all $i \in I$. Let $x \in H$ and let (Q, X) be the unique Leray solution of the linear system (8) with initial condition x. Then there exists a continuous-time Markov chain $(\xi_t)_{t \geq 0}$ defined on a probability space $(S, \mathcal{S}, \mathcal{P})$ taking values in $\mathbb{N}$ and with q-matrix Π defined by (30), such that for all $t \geq 0$*

$$\mathbb{E}^Q\left[|X_n(t)|^2\right] = \|x\|^2 \mathcal{P}(\xi_t = n), \qquad \forall n \geq 1 \tag{41}$$

$$\mathbb{E}^Q\left[\|X(t)\|^2\right] = \|x\|^2 \mathcal{P}(\xi_t \in \mathbb{N}) = \|x\|^2 \mathcal{P}(\tau > t) \tag{42}$$

where

$$\tau := \sup\{t : \xi \text{ has finitely many jumps in } [0,t)\} \in (0, \infty]$$

is the so-called explosion time of the Markov chain.

Proof. Let $p_n^0 := |x_n|^2/\|x\|^2$ for all $n \geq 1$. It is standard to formally construct a continuous-time Markov chain ξ_t on $(S, \mathcal{S}, \mathcal{P})$ with initial distribution p^0 and rates $\pi_{n,m}$ as defined in (30). Heuristically, the process starts at a random position ξ_0 with $\mathcal{P}(\xi_0 = n) = p_n^0$. Then every time the process arrives in a position n it waits for an exponentially distributed random time with rate $\pi_n = \sum_{m\neq n} \pi_{n,m}$ and then jumps to a new random position different from n cheosen with probabilities $\left(\frac{\pi_{n,m}}{\pi_n}\right)_{m\neq n}$. This defines ξ_t up to time τ. At time τ we say that ξ has reached the boundary. (Sometimes this is done by adding one absorbing point θ to the state space.)

For $n \geq 1$, $t \geq 0$, let $p_n(t) := \mathcal{P}(\xi_t = n)$. Then p is a non-negative solution of

$$\begin{cases} p' = p\Pi \\ p(0) = p^0 \end{cases}$$

in $L^\infty([0,\infty); l^1)$. By Corollary 9, $u/\|x\|^2$ is another such solution, hence by the uniqueness result in Lemma 11 we have proved (41) and by summing up also (42). □

The following is the main result for the anomalous dissipation of the auxiliary linear system. The exponential decay of the expected value of energy follows from the Markov property of the chain.

Theorem 5. *Suppose L_i is the identity matrix for all $i \in I$. Let $x \in H$ and let (Q, X) be the unique Leray solution of the linear system (8) with initial condition x. Then the quantity $\mathbb{E}^Q\left[\|X(t)\|^2\right]$ is strictly decreasing in t. Moreover there exists a constant $\mu > 0$ depending only on the coefficients $k_{i,n}$ and a constant $C \geq \|x\|^2$ depending only on $k_{i,n}$'s and x, such that for all $t \geq 0$*

$$\mathbb{E}^Q\left[\|X(t)\|^2\right] \leq Ce^{-\frac{\sigma^2}{\mu}t}$$

Proof. By Proposition 14 we are given a probability space $(S, \mathcal{S}, \mathcal{P})$ and a continuous time Markov chain ξ on the positive integers, defined up to some stopping time τ such that $\mathbb{E}^Q[\|X(t)\|^2] = \|x\|^2\mathcal{P}(\tau > t)$, so we study the latter probability.

Once we will prove the second statement, the fact that $\mathcal{P}(\tau > t)$ is strictly decreasing in t will follow from $\mathcal{P}(\tau = \infty) < 1$ by Chapman-Kolmogorv equation. (This is a standard exercise on continuous-time Markov chains whose proof is not difficult. See for example Lemma 13 in Barbato, Flandoli and Morandin [7].)

We want to prove the exponential bound. Let ζ_k for $k = 0, 1, 2, \dots$ be the *discrete* time Markov chain embedded in ξ, meaning that $\zeta_k = \xi_t$ for t between the k-th and the $(k+1)$-th times of jump of ξ.

For $n \geq 1$, let $V_n := \sharp\{k \geq 1 : \zeta_k = n\}$ be the number of times ζ_k visits n. The law of V_n, conditioned on $V_n \neq 0$ is geometrically distributed. Since the sum of a geometrically distributed number of i.i.d. exponential r.v.'s is exponentially distributed, the total time T_n spent by ξ_t on n, conditioned on ever reaching that site, is exponentially distributed. For $n \geq 1$ we define

$$T_n := \mathcal{L}\{t \geq 0 : \xi_t = n\}$$
$$\nu_n := \mathbb{E}^{\mathcal{P}}[T_n | T_n > 0] = E^{\mathcal{P}}[V_n | V_n > 0]\pi_n^{-1}$$

so that $\tau = \sum_{n\geq 1} T_n$ and for all $t \geq 0$

$$\mathcal{P}(T_n > t) \leq \mathcal{P}(T_n > t | T_n > 0) = e^{-t/\nu_n}$$

We claim that $E^{\mathcal{P}}[V_n | V_n > 0]$ converges to some finite limit as $n \to \infty$.

Then, since by Remark 8, $\pi_n = \sigma^2\lambda^{2n}\sum_{i\in I_n} k_i^2 = O(\lambda^{2n})$, we have $\nu_n = O(\lambda^{-2n})$, so that the quantities $\nu := \sum_{n\geq 1}\nu_n$ and $\Lambda := -\sum_{n\geq 1}\nu_n \log \nu_n$ are

both finite. Define the sequence of numbers $(\theta_n)_{n\geq 1}$ satisfying

$$e^{-t\theta_n/\nu_n} = \nu_n e^{(\Lambda-t)/\nu}, \qquad n \geq 1$$

and notice that

$$\sum_{n\geq 1} \theta_n = \frac{1}{t}\sum_{n\geq 1}\Big[-\nu_n \log \nu_n - (\Lambda - t)\frac{\nu_n}{\nu}\Big] = 1$$

so we conclude that

$$\mathcal{P}(\tau > t) \leq \mathcal{P}\bigg(\bigcup_{n\geq 1}\{T_n > \theta_n t\}\bigg) \leq \sum_{n\geq 1}\mathcal{P}(T_n > \theta_n t) \leq \sum_{n\geq 1} e^{-t\theta_n/\nu_n} = \nu e^{(\Lambda-t)/\nu}$$

This proves the theorem for

$$C = \|x\|^2 \nu e^{\Lambda/\nu} = \|x\|^2 \exp\Big\{-\sum_{n\geq 1}\frac{\nu_n}{\nu}\log\frac{\nu_n}{\nu}\Big\} \geq \|x\|^2$$

$$\mu = \sigma^2 \nu = \sum_{n\geq 1}\frac{E^{\mathcal{P}}[V_n|V_n > 0]}{\lambda^{2n}\sum_{i\in I_n} k_i^2}$$

We check that these do not depend on σ. From the definitions of ν_n and ν, it is enough to show that the law of ζ_k does not depend on σ.

The transition probabilities of ζ are given by $p_{n,n} = 0$ and $p_{n,m} := \frac{\pi_{n,m}}{\pi_n}$ for $n \neq m$. Recall from (30) that

$$\pi_{n,m} = \sigma^2\lambda^{2n}\sum_{\substack{i\in I_n \\ r_i = m-n}} k_i^2 \qquad \text{and} \qquad \pi_n = \sigma^2\lambda^{2n}\sum_{i\in I_n} k_i^2 \tag{43}$$

meaning in particular that $p_{n,m}$ does not depend on σ.

Finally, we must prove the claim.

Consider $p_{n,n+r}$ and notice that again by (43) and Remark 8, it does not depend on n, for $n \geq n_0$. This means that, as long as $\zeta_k \geq n_0$, ζ behaves like a random walk with increment distribution

$$q_r := \frac{\sum_{i:r_i=r} k_i^2}{\sum_{j\in I} k_j^2}, \qquad r \in \mathbb{Z}.$$

Let ρ_k, for $k = 0, 1, 2, \dots$ be a random walk on $\mathbb{Z}$ defined on $(S, \mathcal{S}, \mathcal{P})$ starting from $\rho_0 = \zeta_0$, with increment distribution q. Since

$$\sum_{j\in I} k_j^2 q_{-r} = \sum_{\substack{i\in I \\ r_i=-r}} k_i^2 = \sum_{\substack{i\in I \\ r_i=r}} k_i^2\lambda^{2r_i} = \sum_{j\in I} k_j^2 q_r \lambda^{-2r}$$

and $\lambda > 1$, we have that $q_{-r} < q_r$ whenever $r > 0$, so ρ has a positive drift.

Now we forget for a moment the starting distribution of ζ and consider only transition probabilities. Let $H = \{\rho_k \geq n_0, \forall k \geq 0\}$ and $K = \{\zeta_k \geq n_0, \forall k \geq 0\}$ and let $n \geq n_0$. Then

$$\mathcal{P}(K|\zeta_0 = n) = \mathcal{P}(H|\rho_0 = n) \tag{44}$$

$$\mathcal{P}(\zeta_k \neq n, \forall n \geq 1|\zeta_0 = n, K) = \mathcal{P}(\rho_k \neq n, \forall n \geq 1|\rho_0 = n, H) \tag{45}$$

Take the limit for $n \to \infty$. Since ρ is a random walk with a positive drift, then (44) converges to 1. Hence in the limit we can drop H, K from (45) and conclude that

$$\lim_{n\to\infty}\mathcal{P}(\zeta_k \neq n, \forall n \geq 1|\zeta_0 = n) = \lim_{n\to\infty}\mathcal{P}(\rho_k \neq n, \forall n \geq 1|\rho_0 = n) > 0$$

This in particular means that $E^{\mathcal{P}}[V_n|V_n > 0]$ converges to some finite limit, which was the claim we had to prove. $\square$

The statement of Theorem 5 is about expectations, but since the decay is at least exponential, it can be refined to an almost sure convergence by virtue of Borel-Cantelli Lemma. Proposition 16 below gives the details.

Lemma 15. *Under the same hypothesis of Theorem 5, let $s \geq 0$, then Q-a.s.*

$$\sup_{t\geq s} \|X(t)\| \leq \|X(s)\|$$

Proof. Let $\chi = X(s)$, and consider the restriction of X to the time interval $[s, \infty)$. Then by Theorems 4 and 2 and the ensuing remark, X is the unique strong Leray solution and has the property that Q-almost surely $\|X(t)\| \leq \|\chi\|$. □

Proposition 16. *Under the same hypothesis of Theorem 5, the total energy of the solution goes to zero at least exponentially fast pathwise under Q,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\|^2 \leq -\frac{\sigma^2}{\mu}, \qquad Q\text{-a.s.}$$

Proof. Let $\alpha > 0$. By Theorem 5 we can bound the probabilities

$$Q(\|X(n)\|^2 > e^{-\alpha n}) \leq Ce^{-n\sigma^2/\mu} e^{\alpha n}, \qquad n \geq 0$$

If we take $\alpha < \sigma^2/\mu$, by Borel-Cantelli lemma there exists a r.v. M, such that Q-a.s. and for all $n \geq 0$ we have $\|X(n)\|^2 \leq Me^{-\alpha n}$.

For $n = 0, 1, \dots$, apply Lemma 15 with $s = n$ to get that Q-a.s.

$$\sup_{t\in[n,n+1)} \|X(t)\|^2 \leq Me^{-\alpha n}$$

or

$$\sup_{t\in[n,n+1)} \|X(t)\|^2 e^{\alpha t-\alpha} \leq \sup_{t\in[n,n+1)} \|X(t)\|^2 e^{\alpha n} \leq M$$

These are countably many propositions, so Q-a.s. all of them are true, yielding

$$\sup_{t\geq 0} \|X(t)\|^2 e^{\alpha t} \leq Me^{\alpha}, \qquad Q\text{-a.s.}$$

From here the thesis follows quickly by letting $\alpha \nearrow \sigma^2/\mu$ on the rational numbers. □

To translate the almost sure statement of the above proposition to the initial non-linear problem, one should be able to prove the equivalence of P and Q on $\mathcal{F}_\infty$ (see Remark 6). The following proposition is the key result to prove Novikov condition of Girsanov theorem for $t = \infty$, which is the object of Theorem 6 below. A very similar statement can be found in Barbato, Flandoli and Morandin [7] and the proof, which is almost the same, is given here for completeness.

Proposition 17. *Under the same hypothesis of Theorem 5, let $\mu > 0$ be the constant given there and let $\theta > 0$. If $\theta < \frac{\sigma^2}{\mu\|x\|^2}$, then*

$$\mathbb{E}^Q\left(e^{\theta \int_0^\infty \|X(t)\|^2 dt}\right) < \infty$$

Proof. Let $V := \int_0^\infty \|X(t)\|^2 dt$ and take any $v \geq 0$. Let $u \geq 0$ defined by

$$\|x\|^2 Q(V > v) = Ce^{-u\sigma^2/\mu} \tag{46}$$

where $\mu > 0$ and $C \geq \|x\|^2$ are the constants given by Theorem 5. Then

$$\begin{aligned} vQ(V > v) \leq \mathbb{E}^Q(V; V > v) &= \int_0^\infty \mathbb{E}^Q(\|X(t)\|^2; V > v)dt \\ &\leq \int_0^\infty \min\bigl(\|x\|^2 Q(V > v); \mathbb{E}^Q(\|X(t)\|^2)\bigr)dt \\ &\leq \int_0^u \|x\|^2 Q(V > v)dt + \int_u^\infty Ce^{-t\sigma^2/\mu}dt \\ &\leq u\|x\|^2 Q(V > v) + \mu\sigma^{-2}Ce^{-u\sigma^2/\mu} \\ &= \|x\|^2 Q(V > v)(u + \mu\sigma^{-2}) \end{aligned}$$

where we used Leray property, Theorem 5 and twice equation (46).

If $Q(V > v) = 0$ for some v then V is bounded and we are done. Otherwise we get a lower bound on u which put into (46) gives

$$Q(V > v) = \frac{C}{\|x\|^2}e^{-u\sigma^2/\mu} \leq \frac{Ce^{-1}}{\|x\|^2}\exp\Bigl\{-\frac{\sigma^2}{\mu\|x\|^2}v\Bigr\}$$

yielding $\mathbb{E}^Q(e^{\theta V}) < \infty$ for all $\theta < \frac{\sigma^2}{\mu\|x\|^2}$. □

Theorem 6. *Under the same hypothesis of Theorem 5, let $\mu > 0$ be the constant given there and let $\rho := \frac{\sqrt{\mu|I|}\|x\|}{2\sigma^2}$. If $\rho < 1$, the total energy of the solution goes to zero at least exponentially fast under P, both pathwise and in average,*

$$\limsup_{t\to\infty} \frac{1}{t}\log\|X(t)\|^2 \leq -\frac{\sigma^2}{\mu}, \qquad P\text{-a.s.} \tag{47}$$

$$\limsup_{t\to\infty} \frac{1}{t}\log\mathbb{E}^P\|X(t)\|^2 \leq -\frac{\sigma^2}{\mu}(1-\rho)^2 \tag{48}$$

Moreover P and Q are equivalent on $\mathcal{F}_\infty$.

Proof. Novikov condition (25) can be extended also to the case $T = \infty$, and Proposition 17 applied with $\theta = \frac{|I^*|}{2\sigma^2} = \frac{|I|}{4\sigma^2}$ shows that it holds on (Ω, Q), so $P \ll Q$. The density is given by (22) with $t = \infty$; it is a.s. positive, P and Q are equivalent. The first statement then follows by Proposition 16.

To prove (48) we follow Barbato, Flandoli and Morandin [7]. Fix $t > 0$ and let $f = \frac{dP_t}{dQ_t} = \exp\{\tilde{Z}_t - \frac{1}{2}[\tilde{Z}, \tilde{Z}]_t\}$ (see (22)). Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\mathbb{E}^P\|X(t)\|^2 = \mathbb{E}^Q\bigl(f\|X(t)\|^2\bigr) \leq \mathbb{E}^Q\bigl(f^p\bigr)^{1/p}\mathbb{E}^Q\bigl(\|X(t)\|^{2q}\bigr)^{1/q}$$

We bound the first term by (24) and Girsanov theorem

$$\begin{aligned} \mathbb{E}^Q\bigl(f^p\bigr) &= \mathbb{E}^Q\bigl(\exp\{p\tilde{Z}_t - \frac{p}{2}[\tilde{Z}, \tilde{Z}]_t\}\bigr) \\ &= \mathbb{E}^Q\bigl(\exp\{p\tilde{Z}_t - \frac{1}{2}[p\tilde{Z}, p\tilde{Z}]_t + \frac{p(p-1)}{2}[\tilde{Z}, \tilde{Z}]_t\}\bigr) \\ &\leq \exp\{\frac{p(p-1)}{2}\frac{|I^*|\|x\|^2 t}{\sigma^2}\}\mathbb{E}^Q\bigl(\exp\{p\tilde{Z}_t - \frac{1}{2}[p\tilde{Z}, p\tilde{Z}]_t\}\bigr) \\ &= \exp\{\frac{p(p-1)|I|\|x\|^2 t}{4\sigma^2}\} \end{aligned}$$

We bound the second term by Leray property and Theorem 5,

$$\mathbb{E}^Q\bigl(\|X(t)\|^{2q}\bigr) \leq \|x\|^{2q-2}\mathbb{E}^Q\bigl(\|X(t)\|^2\bigr) \leq \|x\|^{2q-2}Ce^{-\frac{\sigma^2}{\mu}t}$$

Putting together the two bounds above and with some algebraic manipulations we get that for all $p > 1$

$$\log \mathbb{E}^P \|X(t)\|^2 \leq \log \|x\|^2 + \Big(1 - \frac{1}{p}\Big)\Big(p\rho^2 \frac{\sigma^2}{\mu} t - \frac{\sigma^2}{\mu} t + \log \frac{C}{\|x\|^2}\Big)$$

This formula can be optimized on p. The RHS attains its minimum when

$$p^2 = \rho^{-2}(1 - \frac{\mu}{\sigma^2 t} \log \frac{C}{\|x\|^2})$$

If t is large enough and $\rho < 1$, then this gives an acceptable value $p > 1$. By substituting this value of p and letting $t \to \infty$ we get the thesis. □

7. Applications

In this section we apply our general model to two important shell models of turbulence, namely the inviscid versions of GOY model

$$\frac{d}{dt} u_n = ia\lambda_n u^*_{n+1} u^*_{n+2} + ib\lambda_{n-1} u^*_{n-1} u^*_{n+1} + ic\lambda_{n-2} u^*_{n-1} u^*_{n-2}, \quad n \geq 1 \tag{49}$$

and the inviscid version of Sabra model

$$\frac{d}{dt} u_n = ia\lambda_n u^*_{n+1} u_{n+2} + ib\lambda_{n-1} u^*_{n-1} u_{n+1} - ic\lambda_{n-2} u_{n-1} u_{n-2}, \quad n \geq 1 \tag{50}$$

In both models for $n \geq 1$, u_n are complex-valued functions, $\lambda_n = \lambda^n$, $\lambda > 1$, a, b, c are real numbers with $a + b + c = 0$ and we set $\lambda_n = 0$, $u_n = 0$ for $n \leq 0$ for simplicity.

We may add multiplicative noise to both models to fall in two special cases of our general model (1). This must be done according to the initial requirements and it turns out that the proper way to add noise is for the GOY

$$\begin{aligned} du_n = {} & ia\lambda_n u^*_{n+1} u^*_{n+2} dt + ib\lambda_{n-1} u^*_{n-1} u^*_{n+1} dt + ic\lambda_{n-2} u^*_{n-1} u^*_{n-2} dt \\ & + i\tilde{\sigma}\lambda_n u^*_{n+1} \circ dw_n - i\tilde{\sigma}\lambda_{n-1} u^*_{n-1} \circ dw_{n-1} \end{aligned} \tag{51}$$

and for Sabra

$$\begin{aligned} du_n = {} & ia\lambda_n u^*_{n+1} u_{n+2} dt + ib\lambda_{n-1} u^*_{n-1} u_{n+1} dt - ic\lambda_{n-2} u_{n-1} u_{n-2} dt \\ & + i\tilde{\sigma}_1 \lambda_n u^*_{n+1} \circ dw_n - i\tilde{\sigma}_1 \lambda_{n-1} u^*_{n-1} \circ dw_{n-1} \\ & + (i\tilde{\sigma}_2 \lambda_n u^*_{n+1} \circ dw'_n)^* - i\tilde{\sigma}_2 \lambda_{n-1} u_{n-1} \circ dw'_{n-1} \end{aligned} \tag{52}$$

where $\tilde{\sigma}, \tilde{\sigma}_1, \tilde{\sigma}_2$ are positive constants with $\frac{\tilde{\sigma}_1}{\tilde{\sigma}_2} = \frac{\lambda a}{c}$ and $(w_n)_{n\in\mathbb{Z}}$, $(w'_n)_{n\in\mathbb{Z}}$ are two sequences of complex-valued Brownian motions which are all independent.

Definition 4. Given an initial condition $u^0 \in l^2(\mathbb{C})$, a Leray solution of the stochastic GOY system (51) (respectively of the stochastic Sabra system (52)) is a filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$, along with an adapted sequence $(w_n)_{n\in\mathbb{Z}}$ (resp. two adapted sequences $(w_n)_{n\in\mathbb{Z}}$ and $(w'_n)_{n\in\mathbb{Z}}$) of independent complex-valued Brownian motions, and a stochastic process u such that

i. $u = (u_n)_{n\geq 1}$ is a stochastic process on $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, P)$ taking values in $l^2(\mathbb{C})$ with continuous adapted components;

ii. With probability 1, for all $t \geq 0$, $\|u(t)\|_{l^2} \leq \|u^0\|_{l^2}$.

iii. the following integral equation (resp. equation (54)) holds for all $n \geq 1$ and all $t \geq 0$,

$$u_n(t) = u_n^0 + \int_0^t ia\lambda_n u_{n+1}^*(s)u_{n+2}^*(s)ds + \int_0^t ib\lambda_{n-1}u_{n-1}^*(s)u_{n+1}^*(s)ds + \int_0^t ic\lambda_{n-2}u_{n-1}^*(s)u_{n-2}^*(s)ds - \int_0^t \frac{\tilde{\sigma}^2}{2}(\lambda_n^2 + \lambda_{n-1}^2)u_n(s)ds + \int_0^t i\tilde{\sigma}\lambda_n u_{n+1}^*(s)dw_n(s) - \int_0^t i\tilde{\sigma}\lambda_{n-1}u_{n-1}^*(s)dw_{n-1}(s) \tag{53}$$

$$u_n(t) = u_n^0 + \int_0^t ia\lambda_n u_{n+1}^*(s)u_{n+2}(s)ds + \int_0^t ib\lambda_{n-1}u_{n-1}^*(s)u_{n+1}(s)ds - \int_0^t ic\lambda_{n-2}u_{n-1}(s)u_{n-2}(s)ds - \int_0^t \frac{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}{2}(\lambda_n^2 + \lambda_{n-1}^2)u_n(s)ds + \int_0^t i\tilde{\sigma}_1\lambda_n u_{n+1}^*(s)dw_n(s) - \int_0^t i\tilde{\sigma}_1\lambda_{n-1}u_{n-1}^*(s)dw_{n-1}(s) + \int_0^t (i\tilde{\sigma}_2\lambda_n u_{n+1}^*(s)dw_n'(s))^* - \int_0^t i\tilde{\sigma}_2\lambda_{n-1}u_{n-1}(s)dw_{n-1}'(s) \tag{54}$$

Theorem 7. *Given an initial condition $u^0 \in l^2(\mathbb{C})$, there exists a Leray solution (P, u) of the stochastic GOY system which is unique in law. Moreover for all $t > 0$,*

$$P(\|u(t)\|_{l^2} < \|u^0\|_{l^2}) > 0$$

and for all $\epsilon > 0$ there exists $t > 0$ such that

$$P(\|u(t)\|_{l^2} < \epsilon) > 0$$

Finally, if $\|u^0\|_{l^2}$ is sufficiently small, then for $t \to \infty$, $u(t)$ converges to zero at least exponentially fast both almost surely and in $L^2(\Omega; l^2(\mathbb{C}))$.

Proof. All we need to do is rewrite this model in the formalism of our general model (1). Take $d = 2$, let $\phi : \mathbb{C} \to \mathbb{R}^2$ be the obvious isomorphism and let $X_n := \phi(u_n) := (\mathrm{Re}(u_n), \mathrm{Im}(u_n))$ and $x_n = \phi(u^0)$.

The first step is to define a bilinear operator B on $\mathbb{R}^2$ corresponding to $(v, z) \mapsto iv^*z^*$ on $\mathbb{C}$. For $\alpha, \beta, \gamma \in \{1, 2\}$, let

$$B^{\alpha,\beta,\gamma} = \begin{cases} 1/\sqrt{2} & \alpha + \beta + \gamma = 4 \\ -1/\sqrt{2} & \alpha + \beta + \gamma = 6 \\ 0 & \text{otherwise} \end{cases}$$

so that it is easy to chack that for any $v, z \in \mathbb{C}$, $\phi(iv^*z^*) = \sqrt{2}B(\phi(v), \phi(z))$ and that $L = L^{\alpha,\beta} = \sum_{\gamma,\delta} B^{\alpha,\gamma,\delta}B^{\beta,\gamma,\delta}$ is the identity.

The second step is to choose the interactions corresponding to the GOY local range coupling. Since there are three terms, at least two pair of interactions are needed. Let $I = \{1, 2, 3, 4\}$, $\tau = (1\ 3)(2\ 4)$, $I^* = \{1, 2\}$ and for $i \in I$ let $B_i = B$ and

i	r_i	h_i	k_i
1	1	2	$\sqrt{2}a$
2	-1	-2	$\sqrt{2}\lambda^{-2}c$
3	-1	1	$-\sqrt{2}\lambda^{-1}a$
4	1	-1	$-\sqrt{2}\lambda^{-1}c$

It is now easy to check that if we apply ϕ to the sum of the first three integrals appearing in the RHS of equation (53), we simply get

$$\sum_{i\in I}\int_0^t k_{i,n}B_i(X_{n+r_i}(s),X_{n+h_i}(s))ds \tag{55}$$

Finally, let $\sigma := \tilde{\sigma}/\sqrt{a^2+\lambda^{-2}c^2}$, let $W = (W_{i,n})_{i\in I,n\in\mathbb{Z}}$ be a family of 2-dimensional Brownian motions symmetric with respect to τ, let $(w_n)_{n\in\mathbb{Z}}$ be a sequence of independent complex-valued Brownian motions and suppose that the following equation holds for all n,

$$w_n = \frac{aW^1_{1,n+2}-\lambda^{-1}cW^1_{2,n-1}}{\sqrt{a^2+\lambda^{-2}c^2}} - i\frac{aW^2_{1,n+2}-\lambda^{-1}cW^2_{2,n-1}}{\sqrt{a^2+\lambda^{-2}c^2}} \tag{56}$$

Then

$$\begin{aligned}
&\int_0^t i\tilde{\sigma}\lambda_n u^*_{n+1}(s)dw_n(s)\\
&\qquad\qquad = \int_0^t i\sigma\lambda_n u^*_{n+1}adW^1_{1,n+2} - \int_0^t i\sigma\lambda_n u^*_{n+1}\lambda^{-1}cdW^1_{2,n-1}\\
&\qquad\qquad - \int_0^t i\sigma\lambda_n u^*_{n+1}iadW^2_{1,n+2} + \int_0^t i\sigma\lambda_n u^*_{n+1}i\lambda^{-1}cdW^2_{2,n-1}\\
&= \int_0^t i\sigma a\lambda_n u^*_{n+1}(dW^1_{1,n+2}-idW^2_{1,n+2}) - \int_0^t i\sigma\lambda^{-1}c\lambda_n u^*_{n+1}(dW^1_{2,n-1}-idW^2_{2,n-1})
\end{aligned}$$

so that we can compute ϕ applied to the two stochastic integrals appearing in the RHS of equation (53): we obtain

$$\begin{aligned}
&\phi\Big(\int_0^t i\tilde{\sigma}\lambda_n u^*_{n+1}(s)dw_n(s)\Big)\\
&\quad = \int_0^t i\sigma\sqrt{2}a\lambda_n B(X_{n+1},dW_{1,n+2}) - \int_0^t i\sigma\sqrt{2}\lambda^{-1}c\lambda_n B(X_{n+1},dW_{2,n-1})\\
&\qquad\qquad\qquad\qquad\qquad = \sum_{i=1,4}\int_0^t \sigma k_{i,n}B_i(X_{n+r_i},W_{i,n+h_i})
\end{aligned}$$

and analogously

$$\phi\Big(-\int_0^t i\tilde{\sigma}\lambda_{n-1}u^*_{n-1}(s)dw_{n-1}(s)\Big) = \sum_{i=2,3}\int_0^t \sigma k_{i,n}B_i(X_{n+r_i},dW_{i,n+h_i})$$

The last term is

$$\begin{aligned}
\phi\Big(-\int_0^t \frac{\tilde{\sigma}^2}{2}(\lambda_n^2+\lambda_{n-1}^2)u_n(s)ds\Big) &= -\int_0^t \frac{\sigma^2}{2}(a^2+\lambda^{-2}c^2)(1+\lambda^{-2})\lambda^{2n}X_n(s)ds\\
&= -\sum_{i\in I}\int_0^t \frac{\sigma^2}{2}k^2_{i,n}X_n(s)ds
\end{aligned}$$

We have proved that, under the assumption that (56) holds, if we apply ϕ to equation (53), we get equation (9). But of course, given $(W^j_{i,n})_{i\in I,j=1,2}$, then (56) may be taken as a definition of w_n, so existence of a Leray solution follows from Corollary 13. On the other hand, given w, let $\tilde{w}$ be onother sequence of independent complex-valued Brownian motions, independent from w. Then

$$\begin{pmatrix}W_{1,n+2}\\ W_{2,n-1}\end{pmatrix} := \begin{pmatrix}W_{3,n+2}\\ W_{4,n-1}\end{pmatrix} := \frac{1}{\sqrt{a^2+\lambda^{-2}c^2}}\begin{pmatrix}a & -\lambda^{-1}c\\ \lambda^{-1}c & a\end{pmatrix}\begin{pmatrix}\mathrm{Re}(w_n) & \mathrm{Im}(w_n)\\ \mathrm{Re}(\tilde{w}_n) & \mathrm{Im}(\tilde{w}_n)\end{pmatrix}$$

defines the family W according to the requirements and to (56), so uniqueness in law of the Leray solution follows from Theorem 3.

To prove the two inequalities, remember that $\|X(t)\| = \|u(t)\|_{l^2}$ and apply Theorem 5. Fix $t > 0$. Since $\mathbb{E}^Q(\|X(t)\|) < \|x\|$, then $Q(\|X(t)\| < \|x\|) > 0$, so the same holds for P which is equivalent to Q on $\mathcal{F}_t$.

Fix $\epsilon > 0$. Since $\mathbb{E}^Q(\|X(t)\|) \to 0$ as $t \to \infty$, then for t large enough $Q(\|X(t)\| > \epsilon) < 1$, so the same holds for P which is equivalent to Q on $\mathcal{F}_t$.

Finally, to prove the last statement, we apply Theorem 6. If $\|x\| = \|u^0\|_{l^2}$ is small enough, then $\rho < 1$, so by (47) we get that P-a.s. for all $\epsilon > 0$, for t large

$$\|u(t)\|_{l^2} \leq e^{-\frac{1}{2}(\frac{\sigma^2}{\mu}-\epsilon)t}$$

and by (48) we get that for all $\epsilon > 0$, for t large

$$\|u(t)\|^2_{L^2(\Omega;l^2(\mathbb{C}))} = \mathbb{E}^P\|u(t)\|^2_{l^2} \leq e^{-\left(\frac{\sigma^2}{\mu}(1-\rho)^2-\epsilon\right)t} \qquad \square$$

Theorem 8. *Given an initial condition $u^0 \in l^2(\mathbb{C})$, there exists a Leray solution (P, u) of the stochastic Sabra system which is unique in law. Moreover for all $t > 0$,*

$$P(\|u(t)\|_{l^2} < \|u^0\|_{l^2}) > 0$$

and for all $\epsilon > 0$ there exists $t > 0$ such that

$$P(\|u(t)\|_{l^2} < \epsilon) > 0$$

Finally, if $\|u^0\|_{l^2}$ is sufficiently small, then for $t \to \infty$, $u(t)$ converges to zero at least exponentially fast both almost surely and in $L^2(\Omega; l^2(\mathbb{C}))$.

Proof. We follow the same strategy as for Theorem 7, so let d, ϕ, X, x, I, τ, r_i, h_i and k_i be defined like there. We need three new different bilinear operators on $\mathbb{R}^2$ (below on the right) which represent the corresponding bilinear operators on $\mathbb{C}$ associated to the interactions in the Sabra model (below on the left)

$$(v, z) \mapsto iv^*z \qquad B_1^{\alpha,\beta,\gamma} = B_3^{\alpha,\beta,\gamma} = \begin{cases} 0 & \alpha+\beta+\gamma \text{ odd} \\ -1/\sqrt{2} & \alpha=1,\beta=1,\gamma=2 \\ 1/\sqrt{2} & \text{otherwise} \end{cases}$$

$$(v, z) \mapsto -ivz \qquad B_2^{\alpha,\beta,\gamma} = \begin{cases} 0 & \alpha+\beta+\gamma \text{ odd} \\ -1/\sqrt{2} & \alpha=2,\beta=1,\gamma=1 \\ 1/\sqrt{2} & \text{otherwise} \end{cases}$$

$$(v, z) \mapsto ivz^* \qquad B_4^{\alpha,\beta,\gamma} = \begin{cases} 0 & \alpha+\beta+\gamma \text{ odd} \\ -1/\sqrt{2} & \alpha=1,\beta=2,\gamma=1 \\ 1/\sqrt{2} & \text{otherwise} \end{cases}$$

These B_i satisfy (3) and the corresponding L_i's are the identity.

It is immediate to verify that if we apply ϕ to the sum of the first three integrals appearing in the RHS of equation (54), we simply get

$$\sum_{i\in I} \int_0^t k_{i,n} B_i(X_{n+r_i}(s), X_{n+h_i}(s))ds \tag{57}$$

Finally, let $\sigma := \tilde{\sigma}_1/a = \tilde{\sigma}_2/(\lambda^{-1}c)$, let $(w_n)_{n\in\mathbb{Z}}$ and $(w'_n)_{n\in\mathbb{Z}}$ be two sequences of independent complex-valued Brownian motions and let $W = (W_{i,n})_{i\in I,n\in\mathbb{Z}}$ be a family of 2-dimensional Brownian motions symmetric with respect to τ such that P-a.s. $W_{1,n} = W_{3,n} = \phi(w_{n-2})$ and $W_{2,n} = W_{4,n} = \phi(w'_{n+1})$.

Then it also easy to verify that if we apply ϕ to the sum of the four stochastic integrals appearing in the RHS of equation (54), we get

$$\sum_{i\in I}\int_0^t \sigma k_{i,n} B_i(X_{n+r_i}(s), dW_{i,n+h_i}(s)) \tag{58}$$

The last term is

$$\phi\Big(-\int_0^t \frac{\tilde\sigma_1^2+\tilde\sigma_2^2}{2}(\lambda_n^2+\lambda_{n-1}^2)u_n(s)ds\Big)$$
$$= -\int_0^t \frac{\sigma^2}{2}(a^2+\lambda^{-2}c^2)(1+\lambda^{-2})\lambda^{2n}X_n(s)ds = -\sum_{i\in I}\int_0^t \frac{\sigma^2}{2}k_{i,n}^2 X_n(s)ds$$

We have proved that if we apply ϕ to equation (54), we get equation (9). Then one concludes exactly like in Theorem 7. □

Remark 18. The smallness condition on $\|u^0\|_{l^2}$ can be made precise by computing μ as in the proof of Theorem 5. One has only to observe that in this case the discrete-time embedded Markov chain ζ_k is a simple random walk on the positive integers reflected in 1 and with positive drift $\frac{\lambda^2-1}{\lambda^2+1}$ and do some computations. We give only the result and notice that this choice of μ is not believed to be optimal. For both the stochastic GOY and Sabra models, the condition $\rho<1$ is equivalent to

$$\|u^0\|_{l^2} < \sqrt{2}(\lambda-\lambda^{-1})\sqrt{a^2-\lambda^{-2}c^2}\,\sigma^2$$

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